

An Optimal Mixed Reinsurance Contract From Insurer's and Reinsurer's Viewpoints

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Article History

Received : 12 February 2022; Revised : 10 March 2022; Accepted : 26 March 2022; Published : 30 June 2022

To cite this paper

Amir T. Payandeh-Najafabadi & Ali Panahi-Bazaz (2021). An Optimal Mixed Reinsurance Contract from Insurer's Reinsurer's Viewpoints. *Journal of Econometrics and Statistics*. 2(1), 91-112.

A reinsurance contract should address the conflicting interests of the insurer and reinsurer. Most of the existing optimal reinsurance contracts only consider the interests of one party. This article combines the proportional and stop-loss reinsurance contracts and introduces a new reinsurance contract called proportional-stop-loss reinsurance. Using the balanced loss function, unknown parameters of the proportional-stop-loss reinsurance have been estimated such that the expected surplus for both the insurer and reinsurer are maximized. Several characteristics for the new reinsurance are provided.

Keywords: Proportional reinsurance; Stop-loss reinsurance; Expected utility; Bayesian approach; Balanced loss function.

1. Introduction

Designing an optimal reinsurance strategy is an interesting actuarial problem that must balance several conflicting interests. Most of the existing optimal reinsurance strategies only consider the interest of one side. Gerber (1979) showed that excess of loss reinsurance maximizes the adjustment coefficient when the loading coefficient is independent of the type of reinsurance strategy and the reinsurance premium calculation principle used is the expected value principle.

Other authors have reached similar results for reinsurance that favor the insurance company. Khan

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(1961), Arrow (1963; 1974), Beard et al. (1977), Cai & Tan (2007), Cai et al. (2008) and Tan et al. (2011) all represent the perspective of the insurance company. All research represents one side of the reinsurance contract, but because of the nature of reinsurance contracts, both the insurance and reinsurance companies must be represented. Borch (1960) discussed optimal quota-share retention and stop-loss retention to maximize the product of the expected utility functions of two-party profits. Similar results in favor of two parties were developed by Borch (1969), Ignatov et al. (2004), Kaishev & Dimitrova (2006), Dimitrova & Kaishev (2010), and Cai et al. (2013).

Some researchers have achieved a balance between the desirability of the insurance and reinsurance companies by combining different reinsurance strategies. This approach began with Centeno (1985), who combined quota-share and excess of loss reinsurance strategies and defined a new reinsurance strategy. She assumed that the insurance company will pay $\min\{\alpha X, M\}$ for loss X and constant α and M . She estimated α and M by minimizing the coefficient of variation and the skewness of the insurance loss. Centeno & Simoões (1991) determined parameters for a mixture of quota-share and excess of loss reinsurance so that adjustment coefficient R is maximized. Liang and Guo (2011) used the reinsurance strategy proposed by Centeno (1985) and estimated α and M by maximizing the expected exponential utility from terminal wealth.

Gajek & Zagrodny (2000) showed that for a bounded-above reinsurance premium, the reinsurance strategy that minimizes the variance of the retained risk of the insurance company takes the form $(1 - \alpha)(X - M)I_{[M, \infty)}(X)$ as the reinsurance portion of loss X . Kaluszka (2004) derived an optimal reinsurance strategy that is a trade-off for the insurer between decreasing the variance of the retained risk and the expected value of its gain. Guerra & Centeno (2008) provided optimal reinsurance that maximizes the adjustment coefficient of the retained risk by exploring the relationship between the adjustment coefficient and expected wealth exponential utility. Cai et al. (2013) and Fang & Qu (2014) examined the reinsurance strategy of Centeno (1985). They maximized the joint survival probability of both the insurer and reinsurer and derived a class of estimators for the parameters of the reinsurance strategy. Brachetta and Ceci (2021) developed an optimal reinsurance contract that

simultaneously determine the optimal starting time and optimal retention level of the contract. Under the Solvency II framework, Zanotto and Clemente (2021) combined several reinsurance treaties to obtain an optimal reinsurance contract that maximizes both solvency ratio and portfolio performance. Zhang et al. (2021) combined two quota-share and excess-of-loss reinsurance contracts and employed the optimal control technique to designed an optimal per-loss reinsurance strategy for a given health insurance package. Zhu and Yin (2021) employed a distortion risk measure to obtain an optimal reinsurance contract. The optimal time to merge two proportional reinsurance policies, which maximizes the survival probability of both cedent and reinsurer has been determined by Li and Li (2021).

These results and those of other studies may lead one to conclude that optimality for a reinsurance strategy is either finding a strategy between all possible (or constrained) reinsurance strategies or estimating unknown parameters of a given reinsurance strategy. The present article defines optimal reinsurance by estimating unknown parameters α and M as:

$$Y_i = \alpha \min(X_i, M) \tag{1}$$

which is the insurer portion from random claim X_i under a reinsurance strategy. This form of reinsurance strategy is called proportional excess of loss reinsurance and is a version of the reinsurance from Centeno (1985). More precisely, she considered $Y_i = \min\{\alpha X_i, M\}$ as the insurer portion from random claim X_i . Therefore, one may conclude that, there is not any essential difference between the reinsurance strategy (1) and Centeno (1985). But, this article estimates two unknown parameters of the new strategy (1) by taking into account both parties (i.e., insurer's and reinsurer's companies). More precisely, unknown parameters α and M in the proportional excess of loss reinsurance strategy shown in Equation (1) can be estimated in two steps. First, estimate the parameters such that the expected utility of the insurer (or reinsurer) is maximized. Next, use the estimated parameters from the insurer and reinsurer as target estimators. Then, it develops a Bayesian estimator with respect to the doubly-balanced loss function for each parameter so that the expected surplus of the insurer and reinsurer are maximized.

Section 2 defines elements of the proposed method. Section 3 examines the optimal properties of the proportional excess of loss reinsurance strategy. The Bayesian estimator for a doubly-balanced loss function for the parameters of the proportional-excess-loss reinsurance strategy are described in Section 4. Section 5 provides an example of the practical implementation of the results. Section 6 concludes the paper.

2. Preliminaries and Model

Suppose random claim X_i has cumulative distribution function $F(x)$, and survival function $\bar{F}(x)$. Moreover, suppose that random claim X_i can be decomposed to the sum of the insurer portion (Y_i) and reinsurer portion ($I(X_i)$), i.e., $X_i = Y_i + I(X_i)$. Now consider the combination of the proportional and excess of loss reinsurance strategies, such as proportional-excess-loss reinsurance $Y_i = \alpha \min(X_i, M)$.

Next define the value-at-risk (VaR) and tail-value-at-risk (TVaR), the most popular risk measures.

Definition 1. *Suppose X stands for a random risk. The Value-at-Risk and the Tail-Value-at-Risk at level $p \in (0, 1)$, are defined as:*

$$\begin{aligned} \text{VaR}[X; p] &= \inf\{x \in R \mid F_X(x) \geq p\}; \\ \text{TVaR}[X; p] &= \frac{1}{1-p} \int_p^1 \text{VaR}[X; \xi] d\xi, \end{aligned}$$

where $F(x)$ stands for the cumulative distribution function of X .

Random variable X is less dangerous than random variable Y whenever $\text{VaR}[X; \alpha_0] \leq \text{VaR}[Y; \alpha_0]$ for given probability level $\alpha_0 \in (0, 1)$. TVaR is the arithmetic average of the VaRs of X from p to 1. The VaR at a given level p does not provide useful information about the thickness of X , but TVaR does (Denuit et al. 2005). The following represents definition of the ordinary balanced loss function for given target estimators δ_0 and δ_1 , a doubly-balanced loss function. The target estimator is a well-known value for a specific parameter.

Definition 2. Suppose δ_0 and δ_1 are given target estimators for unknown parameter ξ . Moreover, suppose that $\rho(\cdot, \cdot)$ is an arbitrary and given loss function. The doubly-balanced loss function of the measure of closeness of estimator δ to target estimators δ_0 and δ_1 and unknown parameter ξ under loss function $\rho(\cdot, \cdot)$ is

$$\begin{aligned} L_{\rho, \omega_1, \omega_2, \delta_0, \delta_1}(\xi, \delta) = & \omega_1 \rho(\delta_0, \delta) + \omega_2 \rho(\delta_1, \delta) \\ & + (1 - \omega_1 - \omega_2) \rho(\xi, \delta), \end{aligned} \quad (2)$$

where $\omega_1 \in [0, 1)$ and $\omega_2 \in [0, 1)$ are weights which satisfy $\omega_1 + \omega_2 < 1$.

The ordinary balanced loss function with one given target estimator was introduced by Zellner (1994) and improved by Jafari et al. (2006), among others. For convenience, L_0 will subsequently be used instead of $L_{\rho, 0, 0, \delta_0, \delta_1}$ whenever $\omega_1 = 0$ and $\omega_2 = 0$. Theorem (1) derives a Bayesian estimator for ξ under the doubly-balanced loss function $L_{\rho, \omega_1, \omega_2, \delta_0, \delta_1}$.

Theorem 1. Suppose expected posterior losses $\rho(\delta_0, \delta)$ and $\rho(\delta_1, \delta)$ are finite for at least one δ in which $\delta \neq \delta_i$, for $i = 0, 1$. The Bayesian estimator for ξ for prior distribution $\pi(\xi)$ and under $L_{\rho, \omega_1, \omega_2, \delta_0, \delta_1}$ is equivalent to the Bayesian estimator for prior distribution:

$$\begin{aligned} \pi^*(\xi | x) = & \omega_1 1_{\{\delta_0(x)\}}(\xi) + \omega_2 1_{\{\delta_1(x)\}}(\xi) \\ & + (1 - \omega_1 - \omega_2) \pi(\xi | x), \end{aligned}$$

under loss function $L_0 := L_{\rho, 0, 0, \delta_0, \delta_1}$.

Proof. Suppose that measures $\mu_X(\cdot)$ and $\mu'_X(\cdot)$ dominate $\pi(\xi | x)$ and $\pi^*(\xi | x)$, respectively. By the

definition of Bayesian estimators under finite expected posterior loss $\rho(\delta_0, \delta)$ and $\rho(\delta_1, \delta)$:

$$\begin{aligned}
& \arg \min_{\delta} \int_{\Xi} \{\omega_1 \rho(\delta_0, \delta) + \omega_2 \rho(\delta_1, \delta) \\
& \quad + (1 - \omega_1 - \omega_2) \rho(\xi, \delta)\} \pi(\xi | x) d\mu_X(\xi) \\
&= \arg \min_{\delta} \int_{\Xi} \{\omega_1 \rho(\xi, \delta) 1_{\{\delta_0(x)\}}(\xi) + \omega_2 \rho(\xi, \delta) 1_{\{\delta_1(x)\}}(\xi) \\
& \quad + (1 - \omega_1 - \omega_2) \rho(\xi, \delta)\} \pi(\xi | x) d\mu_X(\xi) \\
&= \arg \min_{\delta} \int_{\Xi} \rho(\xi, \delta) \{\omega_1 1_{\{\delta_0(x)\}}(\xi) + \omega_2 1_{\{\delta_1(x)\}}(\xi) \\
& \quad + (1 - \omega_1 - \omega_2)\} \pi(\xi | x) d\mu_X(\xi) \\
&= \arg \min_{\delta} \int_{\Xi \cup \{\delta_0(x)\} \cup \{\delta_1(x)\}} L_0(\xi, \delta) \pi^*(\xi | x) d\mu'_X(\xi) \\
&= \delta^*(x). \quad \square
\end{aligned}$$

This theorem is an extension of Lemma (1) in Jafari et al. (2006). The next lemma provides a Bayesian estimator under the doubly-balanced loss function with square error loss.

Lemma 1. *The Bayesian estimator for prior π and under the doubly-balanced loss function with square error loss ($\rho(\xi, \delta) = (\xi - \delta)^2$) is the square error doubly-balanced loss function given by:*

$$\begin{aligned}
\delta_{\pi, \omega_1, \omega_2}(x) = E_{\pi^*}(\xi | x) &= \omega_1 \delta_0(x) + \omega_2 \delta_1(x) \\
& \quad + (1 - \omega_1 - \omega_2) E_{\pi}(\xi | x). \tag{3}
\end{aligned}$$

Proof. The desired result arrives by the fact that the Bayes estimator under the square error loss is the expectation (in the sense of the Riemann-Stieltjes integral) of the posterior distribution. \square

The Bayesian estimator with respect to the doubly-balanced loss function has two parameters, therefore, one may estimate these two parameter by maximizing the expected surplus of the insurer and reinsurer.

3. Optimal properties of proportional-excess-loss reinsurance

This section considers the proportional-excess-loss reinsurance in Equation (1) and establishes appropriate properties for that reinsurance strategy. Theorem (2) shows that the proportional-excess-loss reinsurance minimizes the variance of the retained risk in some situations.

Theorem 2. *Suppose $I(X)$ and $I_N(X)$ are the reinsurer contribution under an arbitrary reinsurance strategy and the proportional-excess-loss reinsurance for random claim X , respectively. Moreover, suppose that $E(I(X)) = E(I_N(X))$ and*

- (i) $P(I(X) \geq I_N(X) | X \leq M) = 1$;
- (ii) $P(I(X) \geq I_N(X) | X \geq M \& X - I(X) \leq M) = 1$;
- (iii) $P(I(X) \leq I_N(X) | X \geq M \& X - I(X) \geq M) = 1$;

Then variance of the retained risk under the proportional-excess-loss reinsurance is less than such arbitrary reinsurance strategy, i.e., $\text{Var}(X - I(X)) \geq \text{Var}(X - I_N(X))$.

Proof. When $E(I(X)) = E(I_N(X))$, $\text{Var}(X - I(X)) \geq \text{Var}(X - I_N(X))$ whenever $E[(X - I(X))^2] \geq E[(X - I_N(X))^2]$. Setting $W(X) := X - I_N(X) - M$ and $V(X) := X - I(X) - M$. Since $E(W(X)) = E(V(X))$, it suffices to show that $|V(X)| \geq |W(X)|$ with probability one. Now consider the following cases:

- (i) If $X \leq M$ then $W(X) < 0$,

$$\begin{aligned}
 |V(X)| \geq |W(X)| &\Leftrightarrow |X - I(X) - M| \geq M - \alpha X \\
 &\Leftrightarrow M + I(X) - X \geq M - \alpha X \\
 &\Leftrightarrow (1 - \alpha)X \leq I(X) \\
 &\Leftrightarrow I_N(X) \leq I(X);
 \end{aligned}$$

(ii) If $X > M$ then $W(X) < 0$,

$$\begin{aligned}
 |V(X)| &\geq |W(X)| \\
 &\Leftrightarrow |X - I(X) - M| \geq (1 - \alpha)M \\
 &\Leftrightarrow \begin{cases} M - X + I(X) \geq (1 - \alpha)M \text{ for } X - I(X) < M; \\ X - I(X) - M \geq (1 - \alpha)M \text{ for } X - I(X) > M, \end{cases} \\
 &\Leftrightarrow \begin{cases} I(X) \geq I_N(X), & \text{for } X - I(X) < M; \\ I_N(X) \geq I(X), & \text{for } X - I(X) > M, \end{cases} \quad \square
 \end{aligned}$$

Theorem (2) provides conditions under which the variance of the insurer contribution under proportional-excess-loss reinsurance is less than under other reinsurance strategies. Excess of loss and proportional reinsurance strategies do not satisfy Theorem (2) conditions. Therefore, the above finding does not contradict Bowers et al. (1997).

The following theorem compares proportional-excess-loss reinsurance with the proportional reinsurance and the excess of loss reinsurance strategies for stochastic dominance.

Theorem 3. *Suppose $I_N(X)$ is the contribution of reinsurance against random claim X under the proportional-excess-loss reinsurance. Moreover, suppose that $I_P(X)$ ($I_E(X)$) is the contribution of reinsurance against random claim X under the proportional (or the excess of loss) reinsurance strategies. Then:*

$$P(X - I_N(X) \leq X - I_P(X)) = P(X - I_N(X) \leq X - I_E(X)) = 1. \quad (4)$$

Proof. To achieve the desired proof, it suffices to show that $P(A_1) = 1$ ($P(A_2) = 1$), where $A_1 := \{I_N(X) \geq I_E(X)\}$ ($A_2 := \{I_N(X) \geq I_P(X)\}$). Now consider the following two cases:

(i) Under excess of loss reinsurance, $I_E(X) = X - \min(X, M)$; therefore:

$$\begin{aligned}
 P(A_1) &= P(A_1, X \leq M) + P(A_1, X > M) \\
 &= P(0 < X \leq M) + P(X > M) = 1;
 \end{aligned}$$

(ii) Under proportional reinsurance, $I_P(X) = (1 - \alpha)X$; therefore:

$$\begin{aligned} P(A_2) &= P(A_2, X \leq M) + P(A_2, X > M) \\ &= P(X \leq M) + P(X > M) = 1. \quad \square \end{aligned}$$

From Theorem (3) and properties of VaR and the TVaR it can be concluded that the VaR and the TVaR of the insurer contribution under proportional-excess-loss reinsurance is less than for excess of loss and proportional reinsurance strategies, i.e., $\text{VaR}[X - I_N(X); p] \leq \text{VaR}[X - I_E(X); p]$ ($\text{VaR}[X - I_N(X); p] \leq \text{VaR}[X - I_P(X); p]$) for all $p \in (0, 1)$. It can be concluded that $\text{TVaR}[X - I_N(X); p] \leq \text{TVaR}[X - I_E(X); p]$ ($\text{TVaR}[X - I_N(X); p] \leq \text{TVaR}[X - I_P(X); p]$), for all $p \in (0, 1)$.

4. Estimating proportional-excess-loss reinsurance parameters

This section considers proportional excess of loss reinsurance as defined in Equation (1). An optimal reinsurance strategy was derived by estimating unknown parameters α and M . First, the parameters were estimated by maximizing the expected wealth for the insurer (reinsurer) using an exponential utility function. Next, the estimated parameters from the insurer and reinsurer were used as target estimators. A Bayesian estimator was developed for the doubly-balanced loss function for each parameter to maximize the expected exponential utility of terminal wealth for the insurer and reinsurer. Parameters α and M were first estimated using exponential utility function to maximize the expected exponential utility of the reinsurer's terminal wealth. Represent the surplus of the insurer in the proportional excess of loss reinsurance strategy as:

$$\begin{aligned} U_t &= u_0 + (1 + \theta_0)E\left(\sum_{i=1}^{N(t)} Y_i\right) - \sum_{i=1}^{N(t)} Y_i \\ &= u_0 + \pi_0(t) - S(t), \end{aligned} \quad (5)$$

where u_0 is the initial wealth of the insurer, random variable Y_i is the insurer portion of random claim X_i , θ_0 is the safety factor, and $N(t)$ is the Poisson process with intensity λ . The expected

wealth of the insurer under the exponential utility $u(x) = -e^{-\beta_0 x}$ is:

$$E\left(-\exp\left(-\beta_0(U_0 + \pi_0(t) - \sum_{i=1}^{N(t)} Y_i)\right)\right). \quad (6)$$

Using the definition for premium $\pi_0(t)$:

$$\pi_0(t) = (1 + \theta_0)\lambda t \left[\alpha \int_0^M x dF(x) + \alpha M [1 - F(M)] \right], \quad (7)$$

where $f(\cdot)$ and $F(\cdot)$ are the density and distribution functions of random claim X_i , respectively. Theorem (4) provides two estimators for α and M , $\hat{\alpha}_0$ and \hat{M}_0 , that maximize the expected wealth of the insurer Formula (6).

Theorem 4. *Suppose the surplus of the insurer for proportional-excess-loss reinsurance strategy is calculated using Equation (5). Then, $\hat{\alpha}_0$ and \hat{M}_0 maximize the expected exponential utility of the insurer's terminal wealth from Equation (6) as:*

$$\begin{aligned} 0 &= -\hat{\alpha}_0 \beta_0 \hat{M}_0 + \ln(1 + \theta_0), \\ 0 &= -\beta_0(1 + \theta_0)\lambda t \int_0^{\hat{M}_0} x dF(x) - \beta_0(1 + \theta_0)\lambda t \hat{M}_0 \bar{F}(\hat{M}_0) \\ &\quad + \lambda \beta_0 t \int_0^{\hat{M}_0} x e^{\hat{\alpha}_0 \beta_0 x} dF(x) + \lambda \beta_0 t \hat{M}_0 e^{\hat{\alpha}_0 \beta_0 \hat{M}_0} \bar{F}(\hat{M}_0), \end{aligned}$$

where $\bar{F}(\cdot)$ is the survival function.

Proof. Restate the expected exponential utility of the insurer's terminal wealth, (Equation 6) as follows:

$$\begin{aligned} &-e^{-\beta_0(U_0 + \pi_0(t))} E\left(e^{\beta_0 \sum_{i=1}^{N(t)} Y_i}\right) \\ &= -e^{-\beta_0(U_0 + \pi_0(t))} e^{\lambda t (E(e^{\beta_0 Y}) - 1)} \\ &= -e^{(-\beta_0(U_0 + (1 + \theta_0)\lambda t [\alpha \int_0^M x dF(x) + \alpha M \bar{F}(M)])} \\ &\quad \times e^{\lambda t [\int_0^M e^{\alpha \beta_0 x} dF(x) + e^{\alpha \beta_0 M} \bar{F}(M)]}. \end{aligned}$$

Parameters α and M maximize this expression and can be calculated by minimizing:

$$g_0(\alpha, M) = -\beta_0(1 + \theta)\lambda t \left[\alpha \int_0^M x dF(x) + \alpha M \bar{F}(M) \right] + \lambda t \left[\int_0^M e^{\alpha\beta_0 x} dF(x) + e^{\alpha\beta_0 M} \bar{F}(M) \right]. \quad (8)$$

Differentiating $g_0(\alpha, M)$ with respect to α and M and setting them equal to zero produces:

$$\begin{aligned} \frac{\partial g_0}{\partial \alpha} &= -\beta_0(1 + \theta)\lambda t \int_0^M x dF(x) - \beta_0(1 + \theta)\lambda t M \bar{F}(M) \\ &\quad + \lambda \beta_0 t \int_0^M x e^{\alpha\beta_0 x} dF(x) + \lambda \beta_0 t M e^{\alpha\beta_0 M} \bar{F}(M) = 0 \\ \frac{\partial g_0}{\partial M} &= -\beta_0(1 + \theta)\alpha \lambda t \bar{F}(M) + \lambda \alpha \beta_0 t e^{\alpha\beta_0 M} \bar{F}(M) = 0. \end{aligned}$$

It is proven that the solutions to this for α and M , $\hat{\alpha}_0$ and \hat{M}_0 , minimize $g_0(\alpha, M)$. It must be shown that the following Hessian matrix at point $(\hat{\alpha}_0, \hat{M}_0)$ has a positive determinant and that the first argument (a_{11}) is also positive.

$$\begin{pmatrix} \lambda t \int_0^{\hat{M}_0} \beta_0^2 x^2 e^{\hat{\alpha}_0 \beta_0 x} dF(x) + \lambda t \beta_0^2 \hat{M}_0^2 e^{\hat{\alpha}_0 \beta_0 \hat{M}_0} \bar{F}(\hat{M}_0) & \lambda \hat{\alpha}_0 \beta_0^2 t \hat{M}_0 e^{\hat{\alpha}_0 \beta_0 \hat{M}_0} \bar{F}(\hat{M}_0) \\ \lambda \hat{\alpha}_0 \beta_0^2 t \hat{M}_0 e^{\hat{\alpha}_0 \beta_0 \hat{M}_0} \bar{F}(\hat{M}_0) & \lambda \hat{\alpha}_0^2 \beta_0^2 t e^{\hat{\alpha}_0 \beta_0 \hat{M}_0} \bar{F}(\hat{M}_0) \end{pmatrix}.$$

This is arrived at using straightforward calculation. \square

When $\hat{\alpha}_0 > 1 (< 0)$, it must be projected into $[0, 1]$. Now estimate unknown parameters α and M in the proportional-excess-loss reinsurance strategy Equation (1) to maximize the expected exponential utility function ($u(x) = -e^{-\beta_1 x}$) of the reinsurance wealth. Suppose the surplus of reinsurer company under the proportional-excess-loss reinsurance strategy is:

$$U_t^* = u_0^* + \pi_1(t) - \sum_{i=1}^{N(t)} I(X_i) \quad (9)$$

where u_0^* is the initial wealth of the reinsurer, random variable $I(X_i)$ represents the reinsurer portion against random claim X_i , $\pi_1(t)$ is premium of the reinsurance strategy in time t , and $N(t)$ is a Poisson process with intensity λ . Under the expectation premium principle with safety factor θ_1 , premium $\pi_1(t)$ can be restated as:

$$(1 + \theta_1)\lambda t \left[(1 - \alpha) \int_0^M x dF(x) + \int_M^\infty (x - \alpha M) dF(x) \right],$$

where $f(\cdot)$ is the density function of random claim X_i . The expectation of reinsurer wealth using exponential utility function $u(x) = -e^{-\beta_1 x}$ is:

$$E(-\exp(-\beta_1(u_0^* + \pi_1(t) - \sum_{i=1}^{N(t)} I(X_i))))). \quad (10)$$

Theorem (5) provides two estimators for α and M , $\hat{\alpha}_1$ and \hat{M}_1 , that maximize Equation (10).

Theorem 5. *Suppose the surplus for a reinsurance company under the proportional-excess-loss reinsurance strategy can be represented by Equation (9). Then, $\hat{\alpha}_1$ and \hat{M}_1 which maximize the expected exponential utility of the reinsurer's terminal wealth given by Equation (10), can be found as:*

$$\begin{aligned} 0 &= - \int_0^{\hat{M}_1} \beta_1 x e^{\beta_1(1-\hat{\alpha}_1)x} dF(x) - \int_{\hat{M}_1}^{\infty} \beta_1 \hat{M}_1 e^{\beta_1(x-\hat{\alpha}_1\hat{M}_1)} dF(x) \\ &+ \beta_1(1+\theta_1) \int_0^{\hat{M}_1} x dF(x) + \beta_1(1+\theta_1) \int_{\hat{M}_1}^{\infty} \hat{M}_1 dF(x) \\ 0 &= - \int_{\hat{M}_1}^{\infty} \beta_1 \hat{\alpha}_1 e^{-\beta_1(x-\hat{\alpha}_1\hat{M}_1)} dF(x) + \beta_1(1+\theta)\hat{\alpha}_1(1-F(\hat{M}_1)) \end{aligned}$$

Proof. Parameters α and M maximize the expected exponential utility of the reinsurer's terminal wealth in Equation (10) and can be found by minimizing the following expression:

$$\begin{aligned} g_1(\alpha, M) &= \int_0^M e^{\beta_1(1-\alpha)x} dF(x) \\ &+ \int_M^{\infty} e^{\beta_1(x-\alpha M)} dF(x) \\ &- \beta_1(1+\theta_1) \int_0^M (1-\alpha)x dF(x) \\ &- \beta_1(1+\theta_1) \int_M^{\infty} (x-\alpha M) dF(x) \end{aligned}$$

Differentiating $g_1(\alpha, M)$ with respect to α and M and setting them equal to zero produces:

$$\begin{aligned}\frac{\partial g_1}{\partial \alpha} &= - \int_0^M \beta_1 x e^{\beta_1(1-\alpha)x} dF(x) \\ &\quad - \int_M^\infty \beta_1 M e^{\beta_1(x-\alpha M)} dF(x) \\ &\quad + \beta_1(1+\theta) \int_0^M x dF(x) \\ &\quad + \beta_1(1+\theta) \int_M^\infty M dF(x) = 0 \\ \frac{\partial g_1}{\partial M} &= - \int_M^\infty \beta_1 \alpha e^{-\beta_1(x-\alpha M)} dF(x) \\ &\quad + \beta_1(1+\theta)\alpha(1-F(M)) = 0.\end{aligned}$$

The proof shows that the solutions of this equation for α and M , $\hat{\alpha}_1$ and \hat{M}_1 , minimize $g_1(\alpha, M)$. It must be shown that the following Hessian matrix at point $(\hat{\alpha}_1, \hat{M}_1)$ has a positive determinant and $a_{11} > 0$:

$$H_1(\hat{\alpha}_1, \hat{M}_1) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{aligned}a_{11} &= \int_0^{\hat{M}_1} \beta_1^2 x^2 e^{\beta_1(1-\hat{\alpha}_1)x} dF(x) \\ &\quad + \int_{\hat{M}_1}^\infty \beta_1^2 \hat{M}_1^2 e^{\beta_1(x-\hat{\alpha}_1\hat{M}_1)} dF(x) \\ a_{12} &= a_{21} = (-1 + \beta_1 \hat{\alpha}_1 \hat{M}_1) \int_{\hat{M}_1}^\infty \beta_1 e^{\beta_1(x-\hat{\alpha}_1\hat{M}_1)} dF(x) \\ &\quad + \beta_1(1+\theta)(1-F(\hat{M}_1)) \\ a_{22} &= \int_{\hat{M}_1}^\infty \beta_1^2 \hat{\alpha}_1^2 e^{\beta_1(x-\hat{\alpha}_1\hat{M}_1)} dF(x) \\ &\quad + \beta_1 \hat{\alpha}_1 e^{\beta_1(1-\hat{\alpha}_1)\hat{M}_1} f(\hat{M}_1) - \beta_1 \hat{\alpha}_1(1+\theta)f(\hat{M}_1).\end{aligned}$$

the positivity of the determinant of the Hessian matrix cannot be established and must be verified in practice; however, it is evident that $a_{11} > 0$. \square

Thus far, the optimal reinsurance strategy has been defined for the insurer and reinsurer to integrate the results and define an optimal reinsurance strategy that considers the interests of both parties.

A Bayesian estimator was developed for α and M for the doubly-balanced loss function in Equation (2). Estimators $\hat{\alpha}_0$ and $\hat{\alpha}_1$ (\hat{M}_0 and \hat{M}_1) are target estimators for the doubly-balanced loss function. For convenience, $Z_i = I(X_i)$. Lemma (2) provides a cumulative distribution function and density function for conditional random variable $Z|(\theta, \alpha, M)$.

Lemma 2. *Suppose $X|\theta$ has continuous distribution function $F_{X|\theta}(\cdot)$ and continuous density function $f_{X|\theta}(\cdot)$. Moreover, suppose that $Z_1, \dots, Z_n|(\theta, \alpha, M)$ are a sequence of i.i.d. random variables with common density function $f_{Z|(\theta, \alpha, M)}(\cdot)$. Then, the joint density function of $Z_1, \dots, Z_n|(\theta, \alpha, M)$ can be represented as:*

$$f(z_1, \dots, z_n | \theta, \alpha, M) = \left(\frac{1}{1-\alpha} \right)^{n_1} \prod_{i=1}^{n_1} f_{X|\theta} \left(\frac{z_i}{1-\alpha} \right) \\ \times \prod_{i=n_1+1}^n f_{X|\theta}(z_i + \alpha M),$$

where n_1 is the number of z_i s that is less than or equal to $(1-\alpha)M$.

Proof. For one sample $Z|(\theta, \alpha, M)$ the distribution function is:

$$\begin{aligned} F_{Z|\theta, \alpha, M}(z) &= P(Z \leq z) \\ &= P((1-\alpha)X \leq z, X \leq M) \\ &\quad + P(X - \alpha M \leq z, X > M) \\ &= P(X \leq \min\{\frac{z}{1-\alpha}, M\}) \\ &\quad + P(M < X \leq z + \alpha M) \\ &= F_X(\min\{\frac{z}{1-\alpha}, M\}) \\ &\quad + F_X(z + \alpha M) - F_X(M) \\ &= F_X(\frac{z}{1-\alpha})I_{(-\infty, (1-\alpha)M]}(z) \\ &\quad + F_X(z + \alpha M)I_{((1-\alpha)M, \infty)}(z), \end{aligned}$$

where $I_A(x)$ stands for the indicator function. Differentiating $F(z)$ with respect to z leads to:

$$f_{Z|\theta,\alpha,M}(z) = \frac{1}{1-\alpha} f_X\left(\frac{z}{1-\alpha}\right) I_{(-\infty, (1-\alpha)M]}(z) + f_X(z + \alpha M) I_{((1-\alpha)M, \infty)}(z).$$

Suppose that n_1 ($0 \leq n_1 \leq n$) represents the number of z_i s that is less than or equal to $(1-\alpha)M$. The joint density function for an independent sequence of random variables obtained by multiplying their marginal density functions is the desired proof. \square

Lemma (3) develops the joint posterior distribution for (θ, α, M) given random sample Z_1, \dots, Z_n .

Lemma 3. *Suppose $Z_1, \dots, Z_n | (\theta, \alpha, M)$ are a sequence of i.i.d. random variables with common density function $f_{Z|\theta,\alpha,M}(z)$. Moreover, suppose that $\pi_1(\Theta)$, $\pi_2(\mathcal{A})$, and $\pi_3(\mathcal{M})$ are prior distributions for θ , α , and M , respectively. Then, the joint posterior distribution for $(\theta, \alpha, M | Z_1, \dots, Z_n)$ is:*

$$\frac{\left(\frac{1}{1-\alpha}\right)^{n_1} \prod_{i=1}^{n_1} f_{X|\theta}\left(\frac{z_i}{1-\alpha}\right) \prod_{i=n_1+1}^n f_{X|\theta}(z_i + \alpha M) \pi_1(\theta) \pi_2(\alpha) \pi_3(M)}{\int_M \int_{\mathcal{A}} \int_{\Theta} \left(\frac{1}{1-\alpha}\right)^{n_1} \prod_{i=1}^{n_1} f_{X|\theta}\left(\frac{z_i}{1-\alpha}\right) \prod_{i=n_1+1}^n f_{X|\theta}(z_i + \alpha M) \pi_1(\theta) \pi_2(\alpha) \pi_3(M) d\theta d\alpha dM}$$

where n_1 is the number of z_i s that less than or equal to $(1-\alpha)M$.

Proof. The joint density function of $Z_1, \dots, Z_n | (\theta, \alpha, M)$ plus the prior distributions for θ , α , and M are the desired proof. \square

The marginal density functions for $(\alpha | Z_1, \dots, Z_n)$ and $(M | Z_1, \dots, Z_n)$ are

$$\begin{aligned} \pi(\alpha | Z_1, \dots, Z_n) &= \int_{\Theta} \int_{\mathcal{M}} \pi(\theta, \alpha, M | Z_1, \dots, Z_n) dM d\theta; \\ \pi(M | Z_1, \dots, Z_n) &= \int_{\Theta} \int_{\mathcal{A}} \pi(\theta, \alpha, M | Z_1, \dots, Z_n) d\alpha d\theta. \end{aligned}$$

Theorem (6) provides the Bayesian estimator for α and M for the doubly-balanced loss function in Equation (2).

Theorem 6. Suppose $Z_1, \dots, Z_n | (\theta, \alpha, M)$ are a sequence of i.i.d. random variables with common density function $f_{Z|(\theta, \alpha, M)}(z)$. Moreover, suppose that $\pi_1(\Theta)$, $\pi_2(\mathcal{A})$, and $\pi_3(\mathcal{M})$ are prior distributions for θ , α , and M , respectively. Then, the Bayesian estimators for α and M for the square error doubly-balanced loss function, prior distribution π , and target estimators $\hat{\alpha}_0$, $\hat{\alpha}_1$ and \hat{M}_0 , \hat{M}_1 are

$$\begin{aligned}\hat{\alpha}_{\pi, \omega_1, \omega_2} &= \omega_1 \hat{\alpha}_0 + \omega_2 \hat{\alpha}_1 + (1 - \omega_1 - \omega_2) E_{\pi}(\mathcal{A} | z), \\ \hat{M}_{\pi, \omega_1, \omega_2} &= \omega_1 \hat{M}_0 + \omega_2 \hat{M}_1 + (1 - \omega_1 - \omega_2) E_{\pi}(\mathcal{M} | z).\end{aligned}$$

Proof. The results of Lemma (2), Lemma (3), Theorem (1) and Lemma (1) provide the desired proof. \square

5. Simulation study

This section provides two numerical examples to show how the above findings can be applied in practice. It develops (i) estimators for α and M , $\hat{\alpha}_0$ and \hat{M}_0 , so that insurer wealth is maximized; (ii) estimators for α and M , $\hat{\alpha}_1$ and \hat{M}_1 , so that reinsurer's wealth is maximized; (iii) Bayesian estimators for α and M for the square error doubly-balanced loss function for prior distributions $\alpha \sim \text{Beta}(2, 2)$ and $M \sim \text{Exp}(2)$, and target estimators $\hat{\alpha}_0$, $\hat{\alpha}_1$ and \hat{M}_0 , \hat{M}_1 .

Example 1. Suppose 4.117, 1.434, 0.453, 3.333, 0.456, 0.0637, 0.145, 0.211, 3.618, 5.467 is a random sample generated from an exponential distribution with intensity 1. Moreover, suppose that $\text{Beta}(2, 2)$ and $\text{Exp}(2)$ are prior distribution functions for parameters α and M , respectively.

The following provides practical steps to find the optimal proportional-excess-loss reinsurance strategy.

Step 1: Assuming $\beta_0 = 2$ and $\theta_0 = 0.8$, in Theorem (4), lead to $\hat{\alpha}_0 = 0.27$ and $\hat{M}_0 = 1.08$;

Step 2: Assuming $\beta_1 = 0.2$ and $\theta_1 = 0.3$, in Theorem (5), lead to $\hat{\alpha}_1 = 0.38$ and $\hat{M}_1 = 37.001$ where $\det(H_1(0.38, 37.001)) > 0$;

Step 3: Suppose 0.453, 0.456, 0.0637, 0.145, 0.211 in the random sample are $\leq (1 - \alpha)M$. Moreover, suppose that $Beta(2, 2)$ and $Exp(2)$ are prior distribution functions for α and M , respectively. Application of Lemma (1) leads to the Bayesian estimators for α and M :

$$\begin{aligned} \hat{\alpha}_{\pi, \omega_1, \omega_2} &= 0.27\omega_1 + 0.38\omega_2 + 0.6(1 - \omega_1 - \omega_2) \\ &= 0.6 - 0.33\omega_1 - 0.22\omega_2; \\ \hat{M}_{\pi, \omega_1, \omega_2} &= 1.08\omega_1 + 37.001\omega_2 + 0.78(1 - \omega_1 - \omega_2) \\ &= 0.78 + 0.3\omega_1 + 36.221\omega_2, \end{aligned}$$

where, under boundary conditions ω_1 and ω_2 (i.e., $\omega_1 \ \& \ \omega_2 \in [0, 1]$ and $\omega_1 + \omega_2 \leq 1$), both estimators are positive.

Table 1 shows Bayesian estimators $\hat{\alpha}_{\pi, \omega_1, \omega_2}$ and $\hat{M}_{\pi, \omega_1, \omega_2}$ for different values of ω_1 and ω_2 that satisfy the boundary conditions for ω_1 and ω_2 .

Table 1: Bayes estimators $\hat{\alpha}_{\pi, \omega_1, \omega_2}$ and $\hat{M}_{\pi, \omega_1, \omega_2}$ for some different values of ω_1 and ω_2 .

			Bayes estimator	
ω_1	ω_2	$1 - \omega_1 - \omega_2$	$\hat{\alpha}_{\pi, \omega_1, \omega_2}$	$\hat{M}_{\pi, \omega_1, \omega_2}$
0.1	0.1	0.8	0.545	4.43
0.1	0.2	0.7	0.523	8.05
0.1	0.3	0.6	0.501	11.67
0.1	0.4	0.5	0.479	15.29
0.1	0.5	0.4	0.457	18.92
0.1	0.6	0.3	0.435	22.54
0.1	0.7	0.2	0.413	26.16
0.1	0.8	0.1	0.391	29.78
0.1	0.9	0	0.369	33.40
0.1	0.1	0.8	0.545	4.432
0.2	0.1	0.7	0.512	4.462
0.3	0.1	0.6	0.479	4.492
0.4	0.1	0.5	0.446	4.522
0.5	0.1	0.4	0.413	4.552
0.6	0.1	0.3	0.380	4.582
0.7	0.1	0.2	0.347	4.612
0.8	0.1	0.1	0.314	4.642
0.9	0.1	0	0.281	4.672

As one may observe from result of Table 1, choice of ω_1, ω_2 have a big impact on estimated M and do not have such impact on estimated α .

Using result of Table 1, one may determine the desired optimal proportional-excess-loss reinsurance strategy.

The following example assume $\omega_1 = 0.25, \omega_2 = 0.15$ in Theorem (6) provides the optimal proportional-excess-loss reinsurance strategy for some different claim size distributions.

Example 2. Suppose X_1, \dots, X_{100} is a sequence of random sample from distributions given by the first column of Table 2. Moreover, suppose that, for each claim size distribution, prior distributions for α and M are given by Table 2.

For each claim size distribution, we generate random sample X_1, \dots, X_{100} , 100 times and estimate parameters α and M , for each iteration. Table 2 represents mean (and standard deviation) of Bayes estimator of α and M for such 100 iterations.

To estimate unknown parameters α and M using the Bayesian method, we need initials to categorized data into two groups and prior distribution functions for α and M . Prior distribution functions given by the second and third columns of Table 2.

Table 2: Random claim and prior distributions accompanied with posterior's mean and standard deviation.

Claim size distribution	prior for α	prior for M	The mean (SD) for α	The mean (SD) for M
EXP(1)	Beta(2,2)	EXP(2)	0.5189 (0.03300)	3.6915 (0.04020)
EXP(4)	Beat(2,2)	EXP(2)	0.4306 (0.03960)	0.7456 (0.00002)
EXP(8)	Beta(3,2)	Gamma(2,2)	0.4402 (0.03660)	1.0748 (0.00001)
Weibull(2,1)	Beta(2,4)	Gamma(3,2)	0.5458 (0.01500)	1.3813 (0.00003)
Weibull(4,1)	Beat(5,2)	Gamma(2,4)	0.5464 (0.44160)	0.9142 (0.02340)
Weibull(2,4)	Uniform(0,1)	Gamma(3,4)	0.7612 (0.00780)	2.4772 (0.02340)

The small standard deviation of these estimators shows that the estimation method is an appropriate method to use with the different samples. Moreover, as one may observe from result of Table 2,

choice of prior distributions for α and β have a sufficient impact on estimated M and do not have such impact on estimated α .

The mean represented in Table 2 can be considered as a Bayesian estimator for α and M and determine optimal proportional-excess-loss reinsurance.

6. Conclusion

This study combined excess of loss and proportional reinsurance strategies to introduce a new reinsurance strategy, say proportional-excess-loss reinsurance. This optimal reinsurance strategy has been achieved by estimating unknown parameters for the proportional-excess-loss reinsurance strategy such that the expected exponential utility of the insurer's and reinsurer's terminal wealth are maximized, simultaneously.

This new proportional-excess-loss reinsurance strategy can be extended to situations where there are more than two unknown parameters in the reinsurance strategy. Then, unknown parameters have been estimated from more optimal criteria.

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